Anderson Localization – Looking Forward

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Lecture 1
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Outline

1. Introduction
2. Anderson Model; Anderson Metal and Anderson Insulator
3. Phonon-assisted hoping conductivity
4. Localization beyond the real space. Integrability and chaos.
5. Spectral Statistics and Localization
9. Many-Body Localization and Ergodicity
Absence of Diffusion in Certain Random Lattices

P. W. Anderson
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.
...very few believed it [localization] at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author...

Nobel Lecture
Nobel Lecture, December 8, 1977

Local Moments and Localized States
Part 1.

Introduction and History
Diffusion Equation: \[ \frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0 \]

\[ \langle r^2 \rangle = Dt \]

\( \rho(\vec{r},t) \) Can be density of particles or energy density. It can also be the probability to find a particle at a given point at a given time.

Einstein theory of Brownian motion, 1905

The diffusion equation is valid for any random walk provided that there is no memory (markovian process).
Diffusion Equation
\[ \frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0 \]

Einstein-Sutherland Relation for electric conductivity \( \sigma \)
\[ \sigma = e^2 D \nu \quad \nu \equiv \frac{dn}{d\mu} \]
Density of states

If electrons would be degenerate and form a classical ideal gas
\[ \nu = \frac{n_{tot}}{T} \]

William Sutherland (1859-1911)
Will a fluctuation (wave packet) spread?

\[ \langle r^2 (t_\geq) \rangle > \langle r^2 (t_\leq) \rangle \]

Einstein (1905): Random walk without memory
\[ \frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0 \]
always diffusion

\[ \langle r^2 \rangle = Dt \]

Anderson (1958): For quantum particles
extended states
not always localized states
\[ \langle r^2 \rangle \xrightarrow{t \to \infty} \text{const} \]
Basic Quantum Mechanics:

\[
\left[-\frac{\nabla^2}{2m} + U(r) - \epsilon_F\right] \psi_\alpha(r) = \xi_\alpha \psi_\alpha(r)
\]

Continuous Unbound states
\[|\psi_\alpha(\vec{r})|^2 \xrightarrow{L \to \infty} O(L^{-d})\]
Extended states

Discrete Bound states
\[|\psi_\alpha(\vec{r})|^2 \xrightarrow{\vec{r} \to \infty} O(e^{-|\vec{r}|/\xi})\]
Localized states

\(L\) System size

\(d\) Number of the spatial dimensions

Potential well
Localization of single-particle wave-functions. Continuous limit:

\[
\left[ -\frac{\nabla^2}{2m} + U(r) - \epsilon_F \right] \psi_\alpha(r) = \xi_\alpha \psi_\alpha(r)
\]

Random potential

\[\psi_\alpha(x)\] extended

\[\xi_{loc}\] localized
**Spin Diffusion**

**Microwave**


Localization of Ultrasound


Localization of Light


Localization of cold atoms


What about charge transport?

Problem: electrons interact with each other
Einstein Relation (1905)

\[ \sigma = e^2 D \frac{dn}{d\mu} \]

**Conductivity**

**Density of states**

**Diffusion Constant**

At zero temperature

Extended states - **metal**

\[ G \propto L^{d-2} \]

\[ \text{const} \]

Localized states - **insulator**:

\[ G \mu e^{L/\xi} ; \quad = 0 ; \quad D = 0 \]
Part 2.

Anderson Model; Anderson Metal and Anderson Insulator
Anderson Model

- Lattice - tight binding model
- Onsite energies $\epsilon_i$ - random
- Hopping matrix elements $I_{ij}$

$-W < \epsilon_i < W$
uniformly distributed

$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$

Anderson Transition

$I < I_c$
Insulator
All eigenstates are localized
Localization length $\xi$

$I > I_c$
Metal
There appear states extended all over the whole system
One-dimensional Anderson Model

\[ \hat{H} = \begin{pmatrix} \varepsilon_1 & I & 0 \\ I & \ddots & I \\ 0 & \cdots & I & \varepsilon_N \end{pmatrix} \]
Q: Why arbitrary weak hopping $I$ is not sufficient for the existence of the diffusion?

Einstein (1905): Marcovian (no memory) process $\rightarrow$ diffusion

Quantum mechanics is not marcovian! There is memory in quantum propagation!

Why?

A: Quantum Interference
Quantum mechanics is not markovian.
There is memory in quantum propagation!

Why?

A: Quantum interference

Memory!
Quantum mechanics is not marcovian
There is memory in quantum propagation!
Why?

A: Quantum Interference

\[ \phi = \oint \vec{p} \, d\vec{r} \]

Phase accumulated when traveling along the loop

The particle can go around the loop in two directions

\[ \varphi_1 = \varphi_2 \]

Constructive interference → probability of the return to the origin gets enhanced → quantum corrections reduce the diffusion constant.

Tendency towards localization

WEAK LOCALIZATION
Localization of single-particle wave-functions.
Continuous limit:

\[ -\frac{\nabla^2}{2m} + U(r) - \epsilon_F \psi_\alpha(r) = \xi_\alpha \psi_\alpha(r) \]

\( d=1; \) All states are localized
\( d=2; \) All states are localized
\( d > 2; \) Anderson transition
Two-site Anderson = Two-well potential

Hamiltonian

\[
\hat{H} = \begin{pmatrix}
    \mathcal{E}_1 & I \\
    I & \mathcal{E}_2
\end{pmatrix}
\]

diagonalize

\[
\hat{H} = \begin{pmatrix}
    E_1 & 0 \\
    0 & E_2
\end{pmatrix}
\]

\[
E_2 - E_1 = \sqrt{(\mathcal{E}_2 - \mathcal{E}_1)^2 + I^2}
\]
\[ \hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \]  

\[ \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \]

\[ E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \frac{\varepsilon_2 - \varepsilon_1}{I} \]

\[ \varepsilon_2 - \varepsilon_1 >> I \]

\[ \varepsilon_2 - \varepsilon_1 << I \]

von Neumann & Wigner “noncrossing rule”

Level repulsion

In general, a multiple spectrum in typical families of quadratic forms is observed only for two or more parameters, while in one-parameter families of general form the spectrum is simple for all values of the parameter. Under a change of parameter in the typical one-parameter family the eigenvalues can approach closely, but when they are sufficiently close, it is as if they begin to repel one another. The eigenvalues again diverge, disappointing the person who hoped, by changing the parameter to achieve a multiple spectrum.
\( \hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \) diagonalize \( \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \)

\[ E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \frac{\varepsilon_2 - \varepsilon_1}{I} \quad \frac{\varepsilon_2 - \varepsilon_1}{I} \gg I \quad \frac{\varepsilon_2 - \varepsilon_1}{I} \ll I \]

von Neumann & Wigner “noncrossing rule”
Level repulsion


What about the eigenfunctions?
What about the eigenfunctions?

\[ \phi_1, \varepsilon_1; \phi_2, \varepsilon_2 \quad \Leftarrow \quad \psi_1, E_1; \psi_2, E_2 \]

- **Off-resonance**: Eigenfunctions are close to the original on-site wave functions

\[ |\varepsilon_2 - \varepsilon_1| >> I \]

\[ \psi_{1,2} = \phi_{1,2} + O\left(\frac{I}{|\varepsilon_2 - \varepsilon_1|}\right)\phi_{2,1} \]

- **Resonance**: In both bonding and anti-bonding eigenstates the probability is equally shared between the sites

\[ |\varepsilon_2 - \varepsilon_1| << I \]

\[ \psi_{1,2} \approx \phi_{1,2} \pm \phi_{2,1} \]
Anderson insulator
Few isolated resonances

Anderson metal
Many resonances and they overlap

Transition: Typically each site is in the resonance with some other one
Anderson Model

- Lattice - tight binding model
- Onsite energies $\varepsilon_i$ - random
- Hopping matrix elements $I_{ij}$

$$-W < \varepsilon_i < W$$
uniformly distributed

$$I_{ij} = \begin{cases} I & \text{i and j are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$I < I_c$
Insulator
All eigenstates are localized
Localization length $\xi$

$I > I_c$
Metal
There appear states extended all over the whole system
Condition for Localization:

\[ I < \frac{\text{energy mismatch}}{\# \text{ of n.neighbors}} \]

Energy mismatch:
\[ |\varepsilon_i - \varepsilon_j|_{\text{typ}} = W \]

Number of nearest neighbors:
\[ = 2d \]

Transition:
Typically each site is in the resonance with some other one

A bit more precise:
\[
\frac{I_c}{W} \approx \left( \frac{1}{2d} \right) \left( \frac{1}{\ln d} \right)
\]

Logarithm is due to the resonances, which are not nearest neighbors
Condition for Localization:

\[ \frac{I_c}{W} \approx \left( \frac{1}{2d} \right) \left( \frac{1}{\ln d} \right) \]

Q: Is it correct?

A1: For low dimensions - NO. \( I_c = \infty \) for \( d = 1, 2 \).
   All states are localized. Reason - loop trajectories

A2: Works better for larger dimensions \( d > 2 \).

A3: Is exact on the Bethe lattice
**Condition for Localization:**

\[
\frac{I_c}{W} \sim \left( \frac{1}{2d} \right) \left( \frac{1}{\ln d} \right)
\]

**Q:** Is it correct?

**A1:** For low dimensions - NO. \( I_c = \infty \) for \( d = 1, 2 \)

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**A2:** Works better for larger dimensions \( d > 2 \)

**A3:** Is exact on the Bethe lattice

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**Rule of the thumb**

- At the localization transition a typical site is in resonance with another one

- At the localization transition the hoping matrix element is of the order of the typical energy mismatch divided by the number of nearest neighbors
Anderson’s recipe:

Consider an open system (Anderson Model). Particle escape → “broadening” of each eigenstate

\[ E \rightarrow E + i\Gamma; \quad \Gamma \equiv \text{Im} \Sigma \]

\( \Gamma \) - scape rate (inverse dwell time)

\[ I = 0 \quad \Rightarrow \quad \Gamma = 0 \]

States localized inside the system – small \( \Gamma \)

Extended states – large \( \Gamma \)
Anderson’s recipe:

1. take discrete spectrum $E_\mu$ of $H_0$
2. Add an infinitesimal $\text{Im}$ part $i\eta$ to $E_\mu$
3. Evaluate $\text{Im} \Sigma_\mu$

4. take limit $\eta \to 0$ but only after $N \to \infty$

5. “What we really need to know is the probability distribution of $\text{Im} \Sigma$, not its average…”

limits

1) $N \to \infty$
2) $\eta \to 0$

insulator

metal
Probability Distribution of $\Gamma = \text{Im} \Sigma$

Look for:

$$\lim_{\eta \to +0} \lim_{V \to \infty} P(\Gamma > 0) = \begin{cases} > 0; & \text{metal} \\ 0; & \text{insulator} \end{cases}$$
Anderson Transition

\[ I > I_c \]

\[ I < I_c \]

localized and extended never coexist!

all states are localized

\( E_c \) - mobility edges
Temperature dependence of the conductivity

one-electron picture

\[ \sigma(T \to 0) > 0 \quad \sigma(T) \propto e^{\frac{E_c - \epsilon_F}{T}} \quad \sigma(T) = 0 \quad \forall T \]

- there are extended states \( I > I_c \)
- all states are localized \( I < I_c \)
Part 3.

Localization beyond real space

Integrability and chaos
Chaos, Quantum Recurrences, and Anderson Localization

Shmuel Fishman, D. R. Grempel, and R. E. Prange
Department of Physics and Center for Theoretical Physics, University of Maryland, College Park, Maryland 20742
(Received 6 April 1982)

A periodically kicked quantum rotator is related to the Anderson problem of conduction in a one-dimensional disordered lattice. Classically the second model is always chaotic, while the first is chaotic for some values of the parameters. With use of the Anderson-model result that all states are localized, it is concluded that the local quasienergy spectrum of the rotator problem is discrete and that its wave function is almost periodic in time. This allows one to understand on physical grounds some numerical results recently obtained in the context of the rotator problem.
Quantum and Classical Dynamical Systems

Large number $d \gg 1$ of the degrees of freedom

**Conventional Boltzmann-Gibbs Statistical Physics**
- Equipartition Postulate
- Ergodicity: time average = space (ensemble) average
- Chaos
- Hamiltonian $H(\{p_i, q_i\})$

**Integrable Systems**
- $d$ degrees of freedom
- Integrals of motion
- Ergodicity is violated
- Invariant tori dimension $d$
- Hamiltonian $H_0(\{p_i, q_i\})$
- Energy shell, dimension $2d - 1$

**Classical Dynamics**

$\lambda$

- Ergodicity
- Equipartition

**Quantum Dynamics** ???

Fermi, Pasta, Ulam system
(connected nonlinear oscillators)

Solar system

KAM region

Arnold diffusion
Non-ergodic

$0$
Kolmogorov – Arnold – Moser (KAM) theory

Integrable classical Hamiltonian

\[ \hat{H}_0 \quad d > 1: \]

Separation of variables: \( d \) sets of action-angle variables

\[ I_1, \theta_1 = 2\pi \omega_1 t; \ldots, I_2, \theta_2 = 2\pi \omega_2 t; \ldots \]

Quasiperiodic motion: set of the frequencies, \( \omega_1, \omega_2, \ldots, \omega_d \), which are in general incommensurate. Actions \( I_i \) are integrals of motion \( \partial I_i / \partial t = 0 \)


Andrey Kolmogorov

Vladimir Arnold

Jurgen Moser

tori
Kolmogorov – Arnold – Moser (KAM) theory


Given the set of the integrals of motion \( \{I_\mu\} \) all trajectories belong to a torus

Will an arbitrary weak perturbation \( \hat{V} \) of an integrable Hamiltonian \( \hat{H}_0 \) destroy the tori and make the motion ergodic (when each point at the energy shell will be reached sooner or later)

Q: ?

A: Most of the tori survive weak and smooth enough perturbations

KAM theorem
Classical, $d \gg 1$ degrees of freedom

Integrals of motion are $I_1, I_2, \ldots, I_d$

Quantum, $d \gg 1$ degrees of freedom

Integrals of motion are quantized – quantum numbers
Form a “lattice”
Sites of the “lattice” – eigenstate of the integrable system

Classical Dynamics: from KAM to Chaos

$H = H_0 + \lambda V$

Quantum Dynamics: Many – Body Localization
Two integrals of motion

\[ I_1 = p_x; \quad I_1 = p_y \]

Energy shell:

\[ p_x^2 + p_y^2 = 2mE \]
Classical, $d \gg 1$ degrees of freedom

- Integrals of motion: $I_1, I_2, \ldots, I_d$
- Space of the integrals of motion
- KAM: Most of the tori survive weak and smooth perturbation
- Energy Shell: $\hat{\hbar} = 0$

Quantum, $d \gg 1$ degrees of freedom

- Integrals of motion are quantized - quantum numbers
- Form a “lattice”.
- Sites of the “lattice” - eigenstate of the integrable system
- Perturbation - coupling of the different sites of the “lattice” - bonds

$L = L_0 + \lambda V$
Classical, $d \gg 1$ degrees of freedom

Integrals of motion

$I_1, I_2, \ldots, I_d$

integrals of motion

Space of the integrals of motion

Quantum, $d \gg 1$ degrees of freedom

Integrals of motion are quantized - quantum numbers

Form a “lattice”.

Sites of the “lattice” - eigenstate of the integrable system

Perturbation - coupling of the different sites of the “lattice” - bonds

Wave functions localized in the space of quantum numbers

Classical Dynamics: from KAM to Chaos

$H = H_0 + \lambda V$

Quantum Dynamics: Many - Body Localization

Finite motion
Classical, \( d \gg 1 \) degrees of freedom

- Integrals of motion: \( I_1, I_2, \ldots, I_d \)
- Space of the integrals of motion
- KAM: Most of the tori survive weak and smooth perturbation
- Energy Shell

Quantum, \( d \gg 1 \) degrees of freedom

- Integrals of motion are quantized - quantum numbers
- Form a "lattice".
- Sites of the "lattice" - eigenstate of the integrable system
- Perturbation - coupling of the different sites of the "lattice" - bonds
- Quantum random walk
- Localization?

Many - Body Localization is an analog of the Anderson Localization in a finite-dimensional space of a quantum particle subject to a random potential

\[ H = H_0 + \lambda V \]
Most of the tori survive weak and smooth enough perturbations KAM theorem:

\[ H \neq 0 \]

The integrals of motion are quantized

\[ \psi = \sum_{\mu} c_{\mu} \mu_0 \]

Energy shell

\[ \frac{p_x^2 + p_y^2}{2m} = E \]
Matrix element of the perturbation

\[
| \mu \rangle_0 = | \vec{I}^{(\mu)} \rangle
\]

\[
\vec{I}^{(\mu)} = \{ I_1^{(\mu)}, \ldots, I_d^{(\mu)} \}
\]

AL hops are local - one can distinguish “near” and “far”

KAM perturbation is smooth enough
Localized
momentum space

extended

Localized
real space

Pradhan & Sridar,
PRL, 2000

Square billiard

Sinai billiard

Disordered
localized

Disordered
extended
### Glossary

<table>
<thead>
<tr>
<th>Classical</th>
<th>Quantum</th>
</tr>
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<tbody>
<tr>
<td><strong>Integrable</strong></td>
<td><strong>Integrable</strong></td>
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<tr>
<td>$H_0 = H_0(\bar{I}); \quad \partial \bar{I}/\partial t = 0$</td>
<td>$\hat{H}<em>0 = \sum</em>{\mu} E_\mu</td>
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<tr>
<td><strong>Perturbation</strong></td>
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</tr>
<tr>
<td>$V; \quad \partial \bar{I}/\partial t \neq 0$</td>
<td>$\hat{V} = \sum_{\mu,\nu} V_{\mu,\nu}</td>
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<tr>
<td><strong>KAM</strong></td>
<td>Localized</td>
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<tr>
<td><strong>Ergodic (chaotic)</strong></td>
<td><strong>Extended?</strong></td>
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Question:

What is the reason to speak about localization if we in general do not know the space in which the system is localized?

Need an invariant (basis independent) criterion of the localization.